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The Enumeration Theorems under (Permutation) Group Action and Their Applications to Combinatorial Problems

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We present a short survey for enumerative combinatorial theory and then describe some topics on enumeration of mapping systems. Roughly speaking, I think, the theory is divided into the following three branches, (1) the enumeration theorems under (permutation) group action and their applications, (2) the principle of inclusion-exclusion and the Möbius function on partially ordered sets, and (3) a trial for uniting (1) and (2), in which (1) and (2) are rather established but (3) is not still done. We now have a general discussion on "enumeration" and "characterization" of objects. Let Ω be a nonempty set of objects. Let $\mathcal{A}$ be a set of attributes, mathematically speaking, an abstract set with a given structure. Let $\mathfrak{C}$ be a relation between Ω and $\mathcal{A}$. Then a Galois correspondence

$$\wp(\Omega) \underset{\tilde{\mathcal{E}}}{\overset{\tilde{\mathfrak{C}}}{\rightleftarrows}} \wp(\mathcal{A})$$

is induced by the relation $\mathfrak{C}$ and the inverse relation $\mathcal{E}$ (see Ore [55] for the Galois correspondence induced by a relation). The notation $\mathfrak{C}$ and $\mathcal{E}$ have a suggestive look of "characterization" and "enumeration". The map $\tilde{\mathfrak{C}}$ is closely related to the structure theory and so the map $\tilde{\mathcal{E}}$ to the enumeration theory. More generally speaking, the two maps have a close connection with "analysis" and "synthesis". Also, this correspondence seems to be a set theoretic representation of Hamilton's diagram in classical logics, which shows the relationships between "extension" and "intension" of a concept. Speaking in respect of "enumeration", if $\tilde{\mathcal{E}}(A) = \phi$ then it means that any object with the abstract property A does not exist, if $\tilde{\mathcal{E}}(A) \neq \phi$ then it means that some objects with the property A exist, resulting in a problem of enumerating the objects. Thus, the Galois correspondence induced by a relation is very useful as a framework in enumeration. Let's consider more concretely in connection with our subject.

1. The Enumeration Theorems Under Group Action Let Ω be a nonempty finite set and $\mathcal{A}$ be a permutation group G on Ω. We define a relation $\mathfrak{g}$ between Ω and G in the following : for each x in Ω and α in G, $x\mathfrak{g}\alpha$ if and only if $\alpha(x) = x$. Then, a Galois correspondence

$$\mathfrak{P}(\Omega) \underset{\tilde{\mathfrak{h}}}{\overset{\tilde{\mathfrak{g}}}{\rightleftarrows}} \mathfrak{P}(G)$$

is induced by the relation $\mathfrak{g}$ and the inverse relation $\mathfrak{h}$. We see easily that for each X in $\mathfrak{P}(\Omega)$ $\tilde{\mathfrak{g}}(X)\,(=\bigcap_{x\in X}\tilde{\mathfrak{g}}(x))$ is the invariant group of X and that for each α in G $|\tilde{\mathfrak{h}}(\alpha)|$ is the character of α, that is, $\tilde{\mathfrak{h}}(\alpha) = \{x\in\Omega \mid \alpha(x) = x\}$. Note that $|\tilde{\mathfrak{h}}(\alpha)|$ is equal to the number of cycles of length 1 in α. The following theorem considered one of the fundamental theorems in enumerative combinatorial theory seems to be the origin of a series of enumeration theorems under group action by Frobenius, Redfield, Pólya, De Bruijn, Harary and Palmer [3,17,15,1,7,14].

<u>Theorem</u> (Frobenius). Let Ω/G denote the set of orbits in Ω relative to G and O_x be the orbit containing an element x in Ω. Then the following identities hold,

(1) $|O_x| = |G|/|\tilde{\mathfrak{g}}(x)|$

(2) $|\Omega/G| = \frac{1}{|G|}\sum_{\alpha\in G}|\tilde{\mathfrak{h}}(\alpha)|$.

This theorem shows that the computability of $|\tilde{\mathfrak{g}}(x)|$ and $|\tilde{\mathfrak{h}}(\alpha)|$ is essential in enumeration. The author has vaguely known that Burnside took up this theorem in his textbook[3]. On the other hand, at this symposium he is presicely taught by Professor Peter M. Neumann that the theorem was formulated by Frobenius (from Cauchy through Netto) and Burnside took up the theorem in his textbook. Since Frobenius'es theorem, Redfield [17], Pólya[15], De Bruijn[1,2], Harary and Palmer[6-10,14] fruitfully used the cycle index of a permutation group G as a generating function, and they and others(Davis[4], Read[16], Harrison[11,12], Robinson[14],$\cdots$) applied their methods to enumeration of mapping patterns, chemical structures, graphs and machines. It is worth noticing that Redfield's work was referred by Littlewood[13], Read[16], Foulkes[5] and appreciated by Harary and Palmer[8]. The cycle index (group reduction function by Redfield) of a permutation group is defined as follows.

$$P(G;x_1,\cdots,x_n)\,(\text{or } Z(G)) = \frac{1}{|G|}\sum_{\alpha\in G} x_1^{p_1} x_2^{p_2}\cdots x_n^{p_n},$$

where n is the degree of G and the cycle structure $\mathbf{t}(\alpha)$ (or cycle type) of α is $(p_1,\cdots,p_n)$, that is, p_i is the number of cycles of length i in α.

We now show some examples in illustration of the theorems. Example 1 is due to De Bruijn[1,2], Harary and Palmer[7] (the special case is due to Davis[4]) and it is elementary but essential in enumeration of finite mapping systems such as mapping patterns (schemata), chemical structures, graphs and machines. Example 2 is one which Harary and Palmer[10] showed to illustrate one of Redfield's theorems[17] (later Read[16]). Example 3 is one which the author[44,48] introduced by the motive of simplifying figure-works in enumerating the reducible or irreducible types of finite systems such as mappings, finite automata and sequential machines.

Example 1. Let $\mathcal{F}(S,T)$ denote the set of mappings from a finite set S to a finite set T. Let G_S be a permutation group on S and G_T be a permutation group on T. Then we define $f \simeq g$ for each f and g in $\mathcal{F}(S,T)$ if and only if there are α in G_S and β in G_T such that $\beta(f(s)) = g(\alpha(s))$ for all s in S. The product group $G_S \times G_T$ is easily considered to be a permutation group on $\mathcal{F}(S,T)$. Therefore, for each (α,β) in $G_S\times G_T$ we obtain the following identity.

$$|\tilde{\aleph}(\alpha,\beta)| = \prod_{1\leq i}\Big(\sum_{j|i} jq_j\Big)^{p_i},$$

where $\mathbf{t}(\alpha) = (p_1,\cdots,p_i,\cdots)$ and $\mathbf{t}(\beta) = (q_1,\cdots,q_i,\cdots)$. Thus, the number of equivalence classes under the relation $\simeq$ (orbits in $\mathcal{F}(S,T)$ relative to $G_S\times G_T$, called patterns or schemata) is obtained from Frobenius'es theorem.

Example 2. Let D_n denote the dihedral group of degree n generated by the cycle (1 2$\cdots$n) and the relation (1 n)(2 n-1)$\cdots$. Then the cycle index of D_n is as follows.

$$Z(D_n) = \frac{1}{2}Z(C_n) + \begin{cases} \frac{1}{2}x_1 x_2^{(n-1)/2} & n\text{:odd} \\ \frac{1}{4}(x_2^{n/2} + x_1^2 x_2^{(n-2)/2}) & n\text{:even} \end{cases}$$

where C_n is the cyclic group of degree n generated by the cycle (1 2$\cdots$n). The cycle index of C_n also is as follows.

$$Z(C_n) = \frac{1}{n}\sum_{k|n} \varphi(k)\, x_k^{n/k},$$

where $\varphi(k)$ is the Euler φ-function. The cycle indices of D_n and C_n were shown by Redfield[17]. On the other hand, since the automorphism group of a cycle graph of order n is D_n, by the Redfield's theorem (later by Read[16]) the number of different superpositions of 2 cycle graphs of order n with the same set of unlabeled points is equal to $Z(D_n) \cap Z(D_n)$, where $\cap$ is the cap operation (originally denoted Ω by Redfield). For m (≥ 2) monomials $x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r}$, $x_1^{j_1}x_2^{j_2}\cdots x_r^{j_r}$, $\cdots$, the cap $\cap$ is defined by

$$x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r} \cap x_1^{j_1}x_2^{j_2}\cdots x_r^{j_r} \cap \cdots$$
$$= \begin{cases} \left(\prod_{k=1}^{r} k^{i_k} i_k!\right)^{m-1} & i_k = j_k = \cdots \text{ for all } k \\ 0 & \text{otherwise (also } 0^0 = 1) \end{cases}$$

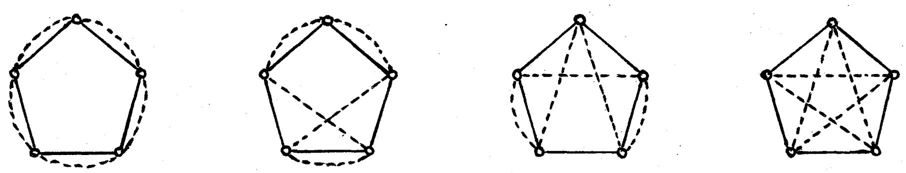
By linearity, the cap operation may be extended to arbitrary polynomials in the variables. For example, for n = 5

$$Z(D_5) = \frac{1}{2}\times\frac{1}{5}(x_1^5 + 4x_5) + \frac{1}{2}x_1x_2^2 = \frac{1}{10}(x_1^5 + 4x_5 + 5x_1x_2^2),$$

$$Z(D_5) \cap Z(D_5) = \frac{1}{100}(x_1^5 \cap x_1^5 + 16x_5 \cap x_5 + 25x_1x_2^2 \cap x_1x_2^2)$$

$$= \frac{1}{100}(120 + 80 + 200) = 4.$$

The four superpositions are as follows :



Example 3. Let PL(S) denote the partition lattice of a n-set S. Let $\mathfrak{S}(S)$ be the symmetric group on S. Then a permutation group $\{\tilde{\alpha}|\alpha \text{ in } \mathfrak{S}(S)\}$ on PL(S), written G(PL(S)), is induced by $\tilde{\alpha}(\pi) = \{\alpha(B)|B\in\pi\}$ for each π in PL(S). Furthermore, a permutation group $\{\tilde{\tilde{\alpha}}|\tilde{\alpha}\in G(PL(S))\}$ on the set C(PL(S)) of chains in PL(S), written G(C(PL(S))), is induced by $\tilde{\tilde{\alpha}}(\mathbf{c}) = \{\tilde{\alpha}(\pi)|\pi\in\mathbf{c}\}$ for each $\mathbf{c}$ in C(PL(S)). In the sequel, we have

$$\mathfrak{S}(S) \cong G(PL(S)) \cong G(C(PL(S)))$$

where $\cong$ denotes "group isomorphic". We now define a cardinal congruence relation $\equiv_{\mathbf{c}}$ on C(PL(S)) in the following : for each $\mathbf{c}$ and $\mathbf{c}'$ in C(PL(S)) $\mathbf{c} \equiv_{\mathbf{c}} \mathbf{c}'$ if and only if there is α in $\mathfrak{S}(S)$ such that $\tilde{\tilde{\alpha}}(\mathbf{c}) = \mathbf{c}'$. Since the cardinal congruence classes under the equivalence relation $\equiv_{\mathbf{c}}$ are the orbits in C(PL(S)) relative to G(C(PL(S))), the following identity is obtained.

$$|O_{\mathbf{c}}| = n!/|G(C(PL(S)))_{\mathbf{c}}|.$$

When $\ell(\mathbf{c})$ (length of $\mathbf{c} = |\mathbf{c}|-1$) = 0, it results in the well known formula, that is, for π in PL(S)

$$|O_\pi| = n!/\prod_{i=1}^{n}(i!)^{p_i}(p_i!),$$

where $\mathbf{t}(\pi)$ (the type of π) = $[(1)^{p_1}(2)^{p_2}\cdots(n)^{p_n}]$ in the set of P(n) of partitions of n. For π in PL(S), $G(PL(S))_\pi$ is equal to the automorphism group of π defined by Ore[53]. We next consider the case of $\ell(\mathbf{c}) = 1$. Let $\gamma(\mathbf{x}) = (p_1,p_2,\cdots,p_k)$ for $\mathbf{x} = [(1)^{p_1}(2)^{p_2}\cdots(k)^{p_k}]$ in P(k). Let π and τ be any elements in PL(S) and $X(\pi) = \{B \text{ in } \pi|B\subseteq X\}$ for X in τ. Let $[\pi,\mathbf{1}]$ be an interval in PL(S) such that $\gamma(\mathbf{t}(\pi)) = v$ and P(v) be the set of partitions of a vector v, where $\mathbf{1}$ is a unique maximal element in PL(S). Then an extended type function $\mathbf{t}_\pi:[\pi,\mathbf{1}]\longrightarrow P(v)$ is well defined by

$$\mathbf{t}_\pi(\tau) = [\gamma(\mathbf{t}(X(\pi)))|X \text{ in } \tau].$$

It is shown that $\pi<\tau \equiv_{\mathbf{c}} \pi'<\tau'$ if and only if $\mathbf{t}_\pi(\tau) = \mathbf{t}_{\pi'}(\tau')$, and the following formula is obtained.

$$|O_{\pi<\tau}| = n!/(\prod_{i=1}^{n}(i!)^{p_i}\prod_{j=1}^{r}(q_j!(\prod_{i=1}^{n}v_{ij}!)^{q_j})),$$

where $\mathbf{t}(\pi) = [(1)^{p_1}\cdots(n)^{p_n}]$, $\mathbf{t}_\pi(\tau) = [v_1^{q_1}\cdots v_r^{q_r}]$ and $v_j = (v_{1j},\cdots,v_{nj})$ for $1\le j\le r$. For example, for

$\pi = \{\overline{1},\overline{2},\overline{3,4},\overline{5,6},\overline{7,8}\}$ and $\tau = \{\overline{1,3,4},\overline{2,5,6},\overline{7,8}\}$,
$t(\pi) = [(1)^2(2)^3]$ and $t_\pi(\tau) = [(1,1)^2(0,1)]$. Then

$$|O_\pi| = 8!/((1!)^2(2!)^3 2!3!) = 420$$

$$|O_{\pi<\tau}| = 8!/((1!)^2(2!)^3 2!(1!1!)^2 1!(0!1!)) = 2520.$$

Furthermore, let $\mathbf{d}$ be a chain in PL(S) and C[$\mathbf{d}$] denote the set of each chain in PL(S) containing $\mathbf{d}$ as a subchain. Then, it is shown that $G(C(PL(S)))_{\mathbf{d}}$ is a permutation group on C[$\mathbf{d}$] and that $\mathbf{c} \equiv_{\mathbf{c}} \mathbf{c}'$ for each $\mathbf{c}$ and $\mathbf{c}'$ in C[$\mathbf{d}$] if and only if there is α in $G(C(PL(S)))_{\mathbf{d}}$ such that $\tilde{\tilde{\alpha}}(\mathbf{c}) = \mathbf{c}'$. Let $O_{\mathbf{c}}^{\mathbf{d}}$ denote the cardinal congruence class (orbit) on C[$\mathbf{d}$] containing $\mathbf{c}$. Then, since for each subchain $\mathbf{d}$ of a chain $\mathbf{c}$ $G(C(PL(S)))_{\mathbf{c}}$ is a subgroup of $G(C(PL(S)))_{\mathbf{d}}$, the following identity is obtained,

$$|O_{\mathbf{c}}^{\mathbf{d}}| = |G(C(PL(S)))_{\mathbf{d}}|/|G(C(PL(S)))_{\mathbf{c}}| = |O_{\mathbf{c}}|/|O_{\mathbf{d}}|.$$

For the previous example π and τ, we have

$$|O_{\pi<\tau}^{\pi}| = |O_{\pi<\tau}|/|O_\pi| = 6.$$

It is noted that the formulas for $\ell(\mathbf{c}) \geq 3$ are open.

2. The Principles of Inclusion-Exclusion on Partially Ordered Sets The enumeration theorems under group action are very useful in counting non-isomorphic types of finite systems, but neverthless they can not answer for the problems of enumerating reducible or irreducible types of finite systems. So, in order to deal with the problems, we have been developing, so called "the enumeration theorems under lattice action", with the Harrison's problem "determine the number of minimal(irreducible) machines with n states" for a background. We now present the fringe of the theory. The principle of inclusion-exclusion on semilattices[45 , Theorem 1] is as follows.

Theorem (Inclusion-Exclusion on Semilattices). Let Ω be a nonempty set and $(L, \vee)$ be a finite join-semilattice. Let $f: L \longrightarrow \wp(\Omega)$ be a map satisfying $f(x) \cap f(y) \subseteq f(x\ y)$ for each x and y in L. Then for any measure m on $\wp(\Omega)$ the following identity holds.

$$m(\bigcup_{x \in L} f(x)) = \sum_{\mathbf{c} \in C} (-1)^{\ell(\mathbf{c})} m(\bigcap_{x \in \mathbf{c}} f(x)),$$

where C is the set of chains in L and $\ell(\mathbf{c})$ denotes the length of a chain $\mathbf{c}$. The theorem can be dualized.

The theorem was applied to a Boolean lattice and a product partition lattice[45, Proposition 1, Theorem 2]. Also, the theorem has been restated in terms of valuations on distributive lattices instead of measures on $\mathfrak{P}(\Omega)$ [46]. The three different proofs have been given, one in which the Rota's theorem from [60, Theorem 1] plays an important role, that is, the closure relation is used, and the others are elemental. Furthermore, recently, the theorem has been extended on partially ordered sets (posets) as follows [49].

<u>Theorem</u> (Inclusion-Exclusion on Posets). Let Ω be a nonempty set and P be a finite partially ordered set with a unique maximal element. Let $f:P \longrightarrow \mathfrak{P}(\Omega)$ be a map satisfying $f(x) \cap f(y) \subseteq f(z)$ for each x and y in P and for some minimal element z in the subposet(of P) of all upper bounds of {x, y}. Then for any measure m on $\mathfrak{P}(\Omega)$ the following identity holds.

$$m\Big(\bigcup_{x \in P} f(x)\Big) = \sum_{\mathbf{c} \in C} (-1)^{\ell(\mathbf{c})} m\Big(\bigcap_{x \in \mathbf{c}} f(x)\Big),$$

where C is the set of all chains in P and $\ell(\mathbf{c})$ denotes the length of a chain $\mathbf{c}$. Also the theorem can be dualized, which results in other three cases. Furthermore, the theorem can be restated in terms of valuations on distributive lattices instead of measures on $\mathfrak{P}(\Omega)$.

We next describe the principle of inclusion-exclusion on partition semilattices and an application of the principle to enumeration of the reducible or irreducible mapping systems. The most simple case is explained (see [48] for the full information). Let $\mathcal{F}(S)$ be the set of mappings from S into S. Let's recall the Galois correspondence $(\tilde{\mathfrak{E}}, \tilde{\mathfrak{E}})$ in Introduction. Then, regarding $\mathcal{F}(S)$ as Ω and PL(S) as $\mathcal{A}$, we define a relation $\mathcal{L}$ between $\mathcal{F}(S)$ and PL(S) regarded the relation $\mathfrak{E}$ in the following way : for each f in $\mathcal{F}(S)$ and π in PL(S) $f\mathcal{L}\pi$ if and only if for each s and t in S $s\pi t$ implies $f(s)\pi f(t)$, where for any x and y in S $x\pi y$ denotes that x and y are contained in a same block of

π. Therefore, a Galois correspondence

$$\wp(\mathcal{F}(S)) \underset{\tilde{\mathcal{S}}}{\overset{\tilde{\mathcal{L}}}{\rightleftarrows}} \wp(PL(S))$$

is induced by the relation $\mathcal{L}$ and the inverse relation $\mathcal{S}$. We see easily that for each F in $\wp(\mathcal{F}(S))$ $\tilde{\mathcal{L}}(F)$ is a sublattice of PL(S) called a <u>reduction diagram</u> of F and that for each π in $\wp(PL(S))$ $\tilde{\mathcal{S}}(\pi)$ is a subsemigroup of the semigroup $\mathcal{F}(S)$ under map composition. Note that the map $\tilde{\mathcal{L}}$ is introduced by making an abstract of a sublattice of partitions with <u>substitution property</u> on a sequential machine studied by Hartmanis and Stearns[37]. Now, the important subset ${}_\tau\mathcal{F}_\pi$ of $\mathcal{F}(S)$ is defined by

$${}_\tau\mathcal{F}_\pi = \{f \in \mathcal{F}(S) \mid \max(\tilde{\mathcal{L}}(f) \cap [\mathbf{0},\tau]) = \pi\},$$

where τ and π are any elements in PL(S) and $\mathbf{0}$ is a unique minimal element in PL(S). Here, $[\mathbf{0},\tau]$ is called a <u>reduction domain</u> of f and $\tilde{\mathcal{L}}(f) \cap [\mathbf{0},\tau]$ is called a <u>reduction diagram of f relative to</u> $[\mathbf{0},\tau]$. In other wards, ${}_\tau\mathcal{F}_\pi$ is the set of f in $\mathcal{F}(S)$ which is at most <u>reduced</u> to $\mathbf{f}:\pi \longrightarrow \pi$ relative to $[\mathbf{0},\tau]$. Therefore, ${}_\tau\mathcal{F}_\mathbf{0}$ is the set of all <u>irreducible mappings relative to</u> $[\mathbf{0},\tau]$. In the computation of $|{}_\tau\mathcal{F}_\pi|$, the following theorem[45, Corollary] is essentially used.

<u>Theorem</u> (Inclusion-Exclusion on Partition Semilattices). Let $\tilde{\mathcal{S}}$ be the map $PL(S) \longrightarrow \wp(\mathcal{F}(S))$ induced by the relation $\mathcal{S}$. Let L be any subsemilattice in PL(S). Then for any measure m on $\wp(\mathcal{F}(S))$ the following identity holds.

$$m(\bigcup_{x \in L} \tilde{\mathcal{S}}(x)) = \sum_{\mathbf{c} \in C} (-1)^{\ell(\mathbf{c})} m(\tilde{\mathcal{S}}(\mathbf{c})),$$

where C is the set of all chains in L.

The map $f:P \longrightarrow \wp(\Omega)$ in the principle of inclusion-exclusion on posets is called a <u>weak morphism</u> on P. In an application of the principle, for a given poset P and map $f:P \longrightarrow \wp(\Omega)$, it is of interest whether f is a weak morphism or not. It is shown that the map $\tilde{\mathcal{S}}$ is a weak morphism on L, and then the theorem follows from the principle. It is also shown that for each chain $\mathbf{c}$ in PL(S) $|\tilde{\mathcal{S}}(\mathbf{c})|$ is characterized by the arithmetic operations. On the other hand, the set ${}_\tau\mathcal{F}_\pi$ is characterized by the map $\tilde{\mathcal{S}}$, and the theorem is applied to compute $|{}_\tau\mathcal{F}_\pi|$. Let's recall the permutation group G(PL(S)) in Example 3. Let α be any element in $\mathfrak{S}(S)$ and $\hat{\alpha}$ be a permutation on $\mathcal{F}(S)$ induced by

$$\hat{\alpha}(f) = \begin{pmatrix} \alpha(1) & \cdots & \alpha(n) \\ \alpha(f(1)) & \cdots & \alpha(f(n)) \end{pmatrix}$$

for each f in $\mathcal{F}(S)$. Then, it is shown that for each α in $\mathfrak{S}(S)$

$$\tilde{\alpha}(\tilde{\mathcal{L}}(f)) = \tilde{\mathcal{L}}(\hat{\alpha}(f)) \text{ and } \hat{\alpha}(_\tau\mathcal{F}_\pi) = {}_{\tilde{\alpha}(\pi)}\mathcal{F}_{\tilde{\alpha}(\tau)}.$$

The present formulation is naturally extended to a relation between the set $(\mathcal{F}(S,T))^p$ of mapping systems and the product partition lattice PL(S)×PL(T), and then the method of counting the number of the reducible or irreducible mapping systems is established by the author[48]. Furthermore, "the relation between $(\mathcal{F}(S))^p$ and PL(S)" and "a relation between machines and PL(S)" are identified by a bijection from the set of transition functions to $(\mathcal{F}(S))^p$. Therefore, the enumeration of reducible or irreducible machines is transformed into the enumeration of reducible or irreducible mapping systems, resulting in the solution for the Harrison's problem. In this enumeration, the cardinal congruence relation described in Example 3 is used to simplify figure-works in it. Furthermore, by considering a permutation group on the set of machines induced by the symmetric group on the set of states, the identity on the number of non-state-isomorphic irreducible machines with n states is also obtained. Finally, it is noted that for each τ in PL(S) $G(PL(S))_\tau$ is a permutation group on $_\tau\mathcal{F}_0$ and that the general computation method for $|_\tau\mathcal{F}_0/G(PL(S))_\tau|$ is not still established.(1) The author thinks that this note is a very simple entrance problem to the third branch described in Introduction.

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(1) Note that $\{\hat{\alpha} \mid \alpha \text{ in } \mathfrak{G}(S)\}$ is a permutation group on $\mathcal{F}(S)$, written $G(\mathcal{F}(S))$, and that we have $G(PL(S)) \cong \mathfrak{G}(S) \cong G(\mathcal{F}(S))$.